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FUNCTION OF MATRICES AND THEIR APPLICATION TO BOUNDARY PROBLEMS FOR A SYSTEM OF DIFFERENTIAL TRANSPORT EQUATIONS

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Аннотация—Методами применения функции от матриц решена краевая задача для системы дифференциальных уравнений переноса при наличии двух взаимосвязанных потоков обобщенных зарядов.

В правых частях системы уравнений переноса введены дополнительные слагаемые, которые зависят линейно от неизвестных потенциалов функций распределения обобщенных зарядов.

NOMENCLATURE

x, y, z,	space co-ordinate;
t,	time;
$\alpha, \beta, \gamma, \tau$,	variables of integration;
A,	square matrix of the second order;
U, f, φ,	column matrices;
λ_K,	eigenvalues (spectra) of matrix A;
E,	single square matrix.

THEORETICAL investigations of transfer phenomena with two interdependent flows of generalized charges inside a conductor depend on solutions of a system of parabolic type differential equations [2]:

$$\left.\begin{aligned} \frac{\partial U_1}{\partial t} &= a_{11}\nabla^2 U_1 + a_{12}\nabla^2 U_2 + b_{11} U_1 + b_{12} U_2 \\ \frac{\partial U_2}{\partial t} &= a_{21}\nabla^2 U_1 + a_{22}\nabla^2 U_2 + b_{21} U_1 + b_{22} U_2 \end{aligned}\right\} \quad (1)$$

or

$$\frac{\partial U_j}{\partial t} = \sum_{K=1}^{2} (a_{jK}\nabla^2 U_K + b_{jK} U_K), \quad (j = 1, 2)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The system of equations (1) describes the differential law of the mutual influence of transient distribution fields of generalized charges (heat, matter, electricity, etc.) on the transport flow of each component. Analytical solutions of system (1) obtained for various boundary conditions enable a thorough study to be made of the transport mechanism of generalized charges at their mutual superpositions.

The development of various methods for solving boundary conditions of system (1) is thus of theoretical and practical interest.

Let a semi-infinite three-dimensional medium with two degrees of freedom be a conductor of generalized charges. It is necessary to find functions $U_1(x, y, z, t)$ and $U_2(x, y, z, t)$ satisfying system (1) and the following second order boundary conditions.

$$U_j(x, y, z, t)\,|_{t=0} = f_j(x, y, z) \quad (2)$$

$$\left.\frac{\partial U_j}{\partial x}\right|_{t=0} = \phi_j(y, z, t) \quad (j = 1, 2) \quad (3)$$

$$(0 \leqslant x < \infty, \quad -\infty < y, \quad z < \infty, \quad t > 0)$$

$$U = \begin{Vmatrix} U_1 \\ U_2 \end{Vmatrix}, \quad A = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}, \quad B = \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}.$$

then system (1) in matrix form takes the form:

$$\frac{\partial U}{\partial t} = A\nabla^2 U + BU \quad (4)$$

For differential equation (4) the initial and boundary conditions will be

$$U(x, y, z, t)\,|_{t=0} = f(x, y, z) \tag{5}$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = \varphi(y, z, t) \tag{6}$$

where

$$f = \begin{Vmatrix} f_1 \\ f_2 \end{Vmatrix}, \quad \varphi = \begin{Vmatrix} \varphi_1 \\ \varphi_2 \end{Vmatrix}$$

Let

$$U(x, y, z, t) = e^{Bt}\, V(x, y, z, t) \tag{7}$$

then equation (4) may be written as:

$$\frac{\partial V}{\partial t} = A\, \nabla^2 V \tag{8}$$

For differential equation (8) in a matrix form the boundary conditions will be

$$V(x, y, z, t)\,|_{t=0} = f(x, y, z) \tag{9}$$

$$\left.\frac{\partial V}{\partial x}\right|_{x=0} = e^{-Bt}\, \varphi(y, z, t) \tag{10}$$

The solution of differential equation (8) at boundary conditions (9)–(10) is known:

$$\left.\begin{aligned} V(x, y, z, t) = {} & \frac{1}{[2\sqrt{(\pi t)}]^3} \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{(A^3)}} \exp\left[-\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4At}\right] \\ & \left[1 + \exp\left(-\frac{\alpha x}{At}\right)\right] f(\alpha, \beta, \gamma)\, d\alpha\, d\beta\, d\gamma - \frac{1}{4\sqrt{(\pi^3)}} \int_0^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{[(t-\tau)^3]}} \frac{1}{\sqrt{(A)}} \\ & \exp\left[-\frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4A(t-\tau)}\right] \exp(-\beta\tau)\, \varphi(\beta, \gamma, \tau)\, d\tau\, d\beta\, d\gamma \end{aligned}\right\} \tag{11}$$

On the basis of formula (7) we get

$$\left.\begin{aligned} U(x, y, z, t) = {} & \frac{1}{[2\sqrt{(\pi t)^3}]} \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{(A^3)}} \exp\left[-\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4At}\right] \\ & \left[1 + \exp\left(-\frac{\alpha x}{At}\right)\right] e^{Bt} f(\alpha, \beta, \gamma)\, d\alpha\, d\beta\, d\gamma - \frac{1}{4\sqrt{(\pi^3)}} \int_0^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{[(t-\tau)^3]}} \frac{1}{\sqrt{(A)}} \\ & \exp\left[-\frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4A(t-\tau)}\right] e^{B(t-\tau)}\, \varphi(\beta, \gamma, \tau)\, d\tau\, d\beta\, d\gamma \end{aligned}\right\} \tag{12}$$

To determine the solution of the basic boundary problem of system (1) at boundary conditions (2)–(3) it is necessary to find elements of the single column matrix on the right-hand side of equation (12).

Let us set up characteristic determinants for matrices A and B:

$$\left.\begin{aligned} |A - \lambda E| &= \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0, \\ |B - \mu E| &= \begin{vmatrix} b_{11} - \mu & b_{12} \\ b_{21} & b_{22} - \mu \end{vmatrix} = 0 \end{aligned}\right\} \tag{13}$$

Let λ_1, λ_2 and μ_1, μ_2 be roots of characteristic equations (13) and assume these numbers to be real and different.

Consider the arbitrary function $F(\lambda)$ and the corresponding function of the matrix $F(A)$. For the real and simple roots λ_1, λ_2 the basic formula for the matrix $F(A)$ assumes the form [3]:

$$F(A) = F(\lambda_1)\, Z_1 + F(\lambda_2)\, Z_2 \tag{14}$$

Matrices Z_1 and Z_2 may be completely specified by defining the matrix A and are independent of the choice of the function $F(\lambda)$.

To define the matrices Z_1 and Z_2 take in sequence $\lambda - \lambda_1$, $\lambda - \lambda_2$ instead of $F(\lambda)$, then we get:

$$\| A - \lambda_2 E \| = (\lambda_1 - \lambda_2) Z_1,$$
$$\| A - \lambda_1 E \| = (\lambda_2 - \lambda_1) Z_2.$$

Introducing values of Z_1 and Z_2 into formula (14) we have:

$$F(A) = \frac{1}{\lambda_1 - \lambda_2} \left\{ \left\| \begin{matrix} a_{11} - \lambda_2 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{matrix} \right\| F(\lambda_1) - \left\| \begin{matrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_1 \end{matrix} \right\| F(\lambda_2) \right\} \quad (15)$$

For the matrix B the arbitrary function $F(B)$ will be represented in a similar way:

$$F(B) = \frac{1}{\mu_1 - \mu_2} \{ \| B - \mu_2 E \| F(\mu_2) - \| B - \mu_1 E \| F(\mu_2) \} \quad (16)$$

In formula (16) assume $F(B) = e^{B(t-\tau)}$ and calculate the product of the matrices $F(B)\phi(\beta, \gamma, \tau)$:

$$e^{B(t-\tau)} \phi(\beta, \gamma, \tau) = e^{B(t-\tau)} \left\| \begin{matrix} \phi_1 \\ \phi_2 \end{matrix} \right\| = \frac{1}{\mu_1 - \mu_2} \left\| \begin{matrix} \phi_1^* \\ \phi_2^* \end{matrix} \right\| \quad (17)$$

where

$$\left. \begin{aligned} \varphi_1^* &= \{(b_{11} - \mu_2) \exp[\mu_1(t-\tau)] - (b_{11} - \mu_1) \exp[\mu_2(t-\tau)]\} \varphi_1 + \\ &\qquad (b_{12} \{\exp[\mu_2(t-\tau)] - \exp[\mu_2(t-\tau)]\}) \varphi_2, \\ \varphi_2^* &= (b_{21} \{\exp[\mu_1(t-\tau)] - \exp[\mu_2(t-\tau)]\}) \varphi_1 + \{(b_{22} - \mu_2) \\ &\qquad \exp[\mu_1(t-\tau)] - (b_{22} - \mu_1) \exp[\mu_2(t-\tau)]\} \varphi_2 \end{aligned} \right\} \quad (18)$$

Let

$$F(A) = \frac{1}{\sqrt{(A)}} \exp\left[- \frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4A(t-\tau)} \right]$$

then, on the basis of formula (15) we have:

$$F(A) \cdot \frac{1}{\mu_1 - \mu_2} \left\| \begin{matrix} \varphi_1^* \\ \varphi_2^* \end{matrix} \right\| = \frac{1}{(\lambda_2 - \lambda_2)(\mu_2 - \mu_2)}$$

$$\left\| \begin{matrix} \sum_{K=1}^{2} \varphi_{1K}^* (\beta, \gamma, \tau) \frac{1}{\sqrt{(\lambda_K)}} \exp\left[- \frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4\lambda_K(t-\tau)} \right] \\ \sum_{K=1}^{2} \varphi_{2K}^* (\beta, \gamma, \tau) \frac{1}{\sqrt{(\lambda_K)}} \exp\left[- \frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4\lambda_K(t-\tau)} \right] \end{matrix} \right\| \quad (19)$$

Where

$$\left. \begin{aligned} \varphi_{11}^* &= (a_{11} - \lambda_2) \varphi_1^* + a_{12} \varphi_2^*, \quad \varphi_{12}^* = - [(a_{11} - \lambda_1) \varphi_1^* + a_{12} \varphi_2^*] \\ \varphi_{21}^* &= a_{21} \varphi_1^* + (a_{22} - \lambda_2) \varphi_2^*, \quad \varphi_{22}^* = - [a_{21} \varphi_1^* + (a_{22} - \lambda_1) \varphi_2^*] \end{aligned} \right\} \quad (20)$$

Thus, formula (19) expresses the matrix integrand in the second term of solution (12). Similarly, the elements of the single column matrix in the first term of formula (12) are determined.

Finally, comparing the matrix elements in both hand sides of equation (12), we obtain:

$$U_j(x, y, z, t) = \frac{1}{(\lambda_1 - \lambda_2)(\mu_1 - \mu_2)\sqrt{(\pi)^3}} \sum_{K=1}^{2} \left\{ \frac{1}{[2\sqrt{(\lambda_K t)}]^3} \int_{\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{jK}^*(\alpha, \beta, \gamma) \right.$$
$$\exp\left[-\frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{4\lambda_K t}\right]\left[1 + \exp\left(\frac{\alpha x}{\lambda_K t}\right)\right] d\alpha\, d\beta\, d\gamma -$$
$$\left. - \frac{1}{4\sqrt{(\lambda_K)}} \int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\varphi_{jK}^*(\beta, \gamma, \tau, t)}{\sqrt{[(t-\tau)^3]}} \exp\left[-\frac{x^2 + (y-\beta)^2 + (z-\gamma)^2}{4\lambda_K(t-\tau)}\right] d\tau\, d\beta\, d\gamma \right\}$$
$$(j = 1, 2) \tag{21}$$

where

$$f_{11}^* = (a_{11} - \lambda_2) f_1^* + a_{12} f_2^*, \quad f_{12}^* = -[(a_{11} - \lambda_1) f_1^* + a_{12} f_2^*],$$
$$f_1^* = [(b_{11} - \mu_2)\exp(\mu_1 t) - (b_{11} - \mu_1)\exp(\mu_2 t)] f_1 + \{b_{12}[\exp(\mu_1 t) - \exp(\mu_2 t)]\} f_2,$$
$$f_2^* = \{b_{21}[\exp(\mu_1 t) - \exp(\mu_2 t)]\} f_1 + [(b_{22} - \mu_2)\exp(\mu_2 t) - (b_{22} - \mu_1)\exp(\mu_2 t)] f_2.$$

Formula (21) gives the solution of system (1) at boundary conditions (2)–(3) when the roots of the characteristic equation of the matrix A are positive and different.

The same solution is valid for the case when the characteristic equation roots are multiple. Moreover, instead of formula (14) the following should be taken:

$$F(A) = F(\lambda_0) Z_1 + \left.\frac{\partial F}{\partial \lambda}\right|_{\lambda=\lambda_0} Z_2 \tag{22}$$

where λ_0 is the multiple root of the characteristic equation of the matrix A, $Z_1 = E$, $Z_2 = \| A - \lambda_0 E \|$.

Assume that transport of generalized charges inside a conductor occurs without mutual superposition, i.e. $a_{12} = a_{21} = 0$, $b_{12} = b_{21} = 0$. The classical solution of a heat-conduction (diffusion) equation can be easily obtained from equation (21) for the generalized charges without superposition.

NOTE

The above method of the function of matrices may be applied to system (1) for semi-infinite conductors (for classical bodies: sphere, cylinder, plate, etc.).

In a later work the boundary problem will be solved for a system of differential equations of molecular transport for the case of a three-dimensional anisotropic conductor of generalized charges.

REFERENCES

1. A. V. Luikov and Y. A. Mikhaylov, *Theory of Energy and Mass Transfer* (Teoriya perenosa energii i veshchestva), Izd. Akad. Nauk BSSR, Minsk (1959).
2. A. V. Luikov and Y. A. Mikhaylov, *Theory of Heat and Mass Transfer* (Teoriya teplo-i massoobmena). Gosenergoizdat, Moscow (1963).
3. F. P. Gantmakher, *Theory of Matrices* (Teoriya matrits). Gostekhizdat, Moscow (1953).
4. M. S. Smirnov, A system of differential equations of the drying process (O sisteme differentsialnykh uravnenii protsessa sushki), *Inzh. Fiz. Zh.* No. 9, 40–44 (1961).
5. P. V. Tsoi, A boundary layer problem for a generalized system of equations of energy and mass transfer (Kraevaya zadacha dlya obobshchennoi sistemy uravnenii perenosa energii i veshchestva), *Inzh. Fiz. Zh.* No. 4, 69–74 (1961).
   P. V. Tsoi, The boundary value problem for a system of differential equations of the parabolic type (Kraevaya zadacha dlya sistemy differentsialnykh uravnenii parabolicheskogo tipa), *Inzh. Fiz. Zh.* No. 12, 61–69 (1961).
6. P. V. Tsoi, Analytical solutions for a set of equations of heat and mass transfer for a semi-bounded medium with different boundary conditions, *Int. J. Heat Mass Transfer* 4, 71–78 (1961).

Abstract—The boundary problem for a system of differential transport equations with two interdependent flows of generalized charges is solved by using methods of the function of matrices.

Additional terms which depend linearly on unknown potentials of distribution functions of generalized charges, are introduced into the right-hand side of the system of transport equations.

Résumé—Le problème aux limites pour un système d'équations différentielles de transport avec deux flux interdépendants de charges généralisées est résolu en utilisant les méthodes des fonctions matricialles.

Des termes additifs qui dépendent linéairement des potentiels inconnus de fonctions de distributions des charges généralisées, sont introduits dans les seconds membres du système d'equations de transport.

Zusammenfassung—Ein Grenzproblem für ein System von differentiellen Transportgleichungen für zwei von einander abhängigen Mengenströmen beliebiger Ladung wird mit Hilfe von Matrizenfunktionen gelöst.

Zusätzliche Ausdrücke, die linear von unbekannten Potentialen der Verteilungsfunktionen der beliebigen Ladungen abhängen, werden auf der rechten Seite des Systems der Transportgleichungen eingeführt.